Topological and arithmetical properties of rational plane curves

GRUPO SINGULAR:*

Enrique ARTAL[1], Jorge CARMONA[2], Jose Ignacio COGOLLUDO[1], Mario ESCARIO[3]
Javier FERNÁNDEZ DE BOBADILLA[4], Ignacio LUENGO[2] and Alejandro MELLE[2]

[1]Departamento de Matemáticas
Universidad de Zaragoza
Campus Plaza de San Francisco s/n
50009 Zaragoza, Spain
artal@unizar.es
jicogo@unizar.es

[2]Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
jcarmona@sip.ucm.es
iluengo@mat.ucm.es
amelle@mat.ucm.es

[3]Departamento de Matemáticas y Computación
c/ Luis Ullua s/n
Universidad de la Rioja
Logroño, Spain
mario.escarioe@dmi.urioja.es

[4]Mathematisch Instituut
Universiteit Utrecht
Postbus 80010
35098 TA Utrecht, The Netherlands
bobadilla@math.uu.nl

Para Enrique con afecto y aprecio.

ABSTRACT

We study properties of arrangements of rational plane curves. The existence of such curves with prescribed topological data has a lot of geometric restrictions. We show how to use the theory of singular complex analytic spaces to obtain such restrictions. We also study topological properties of such arrangements.

2000 Mathematics Subject Classification: 14B05,14J17,14H30, 32S50, 14E15, 32S45, 57M25.
Key words: Surface singularity, rational curve, characteristic variety, fundamental group, $\mathbb{Q}$-homology sphere.

*Partially supported by BFM2001-1488-C02-01/02. Fifth author also supported by the Netherlands Organization for Scientific Research (NWO).

Contents

1. Introduction

The main topic of this survey is the study of rational projective plane curves on the complex two dimensional projective space. We are mostly interested in two aspects: their geometrical and topological properties and their relationship with the theory of singular complex analytic spaces. These two aspects are strongly related.

An irreducible plane curve is said to be rational if its normalization has genus 0. If the degree is greater than 2, then the curve must have singular points. It is important to decide whether a curve with prescribed degrees and topological types exists. A lot has been produced on this subject, mainly in the case of cuspidal curves (i.e., curves *homeomorphic* to $\mathbb{P}^1$).

The simplest rational curve is a line. Arrangements of lines are a main subject in the theory of hyperplane arrangements. An important problem is to decide whether or not a line arrangement with a prescribed intersection pattern exists and, in case it does, to which extend topological properties can be derived from such intersection pattern. We will study this question in 3.3 and 3.4.

A natural generalization of line arrangements is the concept of rational arrangements, i.e., curves such that all their irreducible components are rational. We study some topological properties in §3; in §2 we show the relationship with the theory of surface singularities in $\mathbb{C}^3$. We will show some surprising properties of non-existence which come from arithmetical invariants of the singularities of surfaces. Along the paper we will review and propose some interesting *open problems*.

2. Plane curves and singular complex analytic spaces

2.1. Singularities of complex hypersurfaces

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function and let $(V, 0) := (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ of hypersurface singularity defined by the zero locus of f. The *Milnor fibration* of the holomorphic function f at 0 is the C^∞ locally trivial fibration $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$, where $B_\varepsilon(0)$ is the open ball of radius ε centered at 0, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and $\mathbb{D}^*_\eta$ is the open punctured disk ($0 < \eta \ll \varepsilon \ll 1$). The *Milnor fiber* F of f at 0 is any fiber of $f|$; F is a $2n$-dimensional C^∞ manifold. The *monodromy transformation* $h : F \to F$ is the well-defined (up to isotopy) diffeomorphism of F induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of f at* 0 is the corresponding linear transformation $h_* : H_*(F, \mathbb{C}) \to H_*(F, \mathbb{C})$ on the homology groups.

If $(V, 0)$ defines a germ of isolated hypersurface singularity then $H_j(F, \mathbb{C}) = 0$ except for $j = 0, n$. In particular the non-trivial complex algebraic monodromy will be denoted by $h : H_n(F, \mathbb{C}) \to H_n(F, \mathbb{C})$ and $\Delta(t)$ will denote its characteristic polynomial which has the following well-known properties:

(a) $\Delta(t)$ is a product of cyclotomic polynomials.

(b) If N is the maximal size of the Jordan blocks of h then $N \leq n + 1$.

One of the main tools in singularity theory is the resolution of singularities. Essentially one replaces the singular variety by a non-singular space in such a way that there is an isomorphism outside a dense open subset. Let us introduce some definitions in the hypersurface case. An *embedded resolution* of $(V, 0)$ is a proper analytic map $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^{n+1}, 0)$ from a non-singular complex manifold Y such that:

(i) The analytic subspace $\mathcal{D} := \pi^{-1}(\mathrm{Sing}(V))$ of Y is the union of non-singular n-dimensional manifolds in Y which intersect transversally, that is in a neighborhood of any point of $\mathcal{D}$ there exists a local system of coordinates $y_0, \ldots, y_n$ such that $f \circ \pi(y_0, \ldots, y_n) = y_0^{N_0} \cdots y_n^{N_n}$.

(ii) The map $\pi|_{Y \setminus \mathcal{D}}$ is an analytic isomorphism: $Y \setminus \mathcal{D} \to \mathbb{C}^{n+1} \setminus \mathrm{Sing}(V)$.

Let $\pi : Y \to \mathbb{C}^{n+1}$ be an embedded resolution of the hypersurface V defined by the zero locus of f. Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we set

$$E_J := \bigcap_{j \in J} E_j \text{ and } \check{E}_J := E_J \setminus \bigcup_{j \notin J} E_{J \cup \{j\}}.$$

For each $j \in I$, let us denote by N_j the multiplicity of E_j in the divisor of $f \circ \pi$ and by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorphic $(n+1)$-form in $\mathbb{C}^{n+1}$.

The invariant we are interested in is the *local topological zeta function* $Z_{top,0}(f,s) \in \mathbb{Q}(s)$, which is an analytic (but not topological, see [12]) subtle invariant associated with any germ of an analytic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. This rational function was first introduced by J. Denef and F. Loeser as a sort of limit of the p-adic Igusa zeta function, see [22]. Its former definition was written in terms of any embedded resolution of its zero locus germ $(V,0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ (although it does not depend on any particular resolution). In [23], J. Denef and F. Loeser gave an intrinsic definition of $Z_{top,0}(f,s)$ using arc spaces and the motivic zeta function, see also the Séminaire Bourbaki talk of E. Looijenga [35].

Definition 2.1. The *local topological zeta function* of f is:

$$Z_{top,0}(f,s) := \sum_{J \subset I} \chi(\check{E}_J \cap \pi^{-1}(0)) \prod_{j \in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s),$$

where χ denotes the Euler–Poincaré characteristic.

Each exceptional divisor of an embedded resolution $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^{n+1}, 0)$ of the germ $(V,0)$ gives a candidate pole of the rational function $Z_{top,0}(f,s)$. Nevertheless only a few of them give an actual pole of $Z_{top,0}(f,s)$. There are several conjectures related to the topological zeta functions. We focus our attention on the *Monodromy Conjecture*, see [21, 22].

Local Monodromy Conjecture. *If s_0 is a pole of the topological zeta function $Z_{top,0}(f,s)$ of the local singularity defined by f, then $\exp(2i\pi s_0)$ is an eigenvalue of the local monodromy at some complex point of $f^{-1}(0)$.*

Note that if f defines an isolated hypersurface singularity, then $\exp(2i\pi s_0)$ has to be an eigenvalue of the complex algebraic monodromy of the germ $(f^{-1}(0), 0)$. There are three general problems to consider when trying to prove (or disprove) the conjecture using resolution of singularities:

(P1) Explicit computation of an embedded resolution of the hypersurface $(V,0) \subset (\mathbb{C}^{n+1}, 0)$.

(P2) Elimination of the candidate poles which are not actually poles of $Z_{top,0}(f,s)$.

(P3) Explicit computation of the eigenvalues of the complex algebraic monodromy in terms of the resolution data.

The Monodromy Conjecture, which was first stated for the Igusa zeta function, has been proved for curve singularities by F. Loeser [33]. F. Loeser actually proved a stronger version of the Monodromy Conjecture: *any pole of the topological zeta* *function gives a root of the Bernstein polynomial of the singularity.* The behavior of the topological zeta function for germs of curves is rather well understood once an explicit embedded resolution $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of curve singularities is known, e.g. the minimal one. Basically any irreducible component E of the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ which intersects the total transform $\pi^{-1}(V)$ in at most two points has no contribution to the residue of $Z_{top,0}(f, s)$ at the candidate pole. This was proved in consecutive works by Strauss, Meuser, Igusa and Loeser for Igusa's local zeta function, but the same the proof works for the topological zeta function. W. Veys later gave a much simpler and more conceptual proof of this in [46] and proved in [45] that all other E actually do give poles.

In [14], E. Artal, Pi. Cassou–Nogués, I. Luengo and A. Melle have proved the Monodromy Conjecture for the local Igusa and topological zeta functions of a quasi-ordinary polynomial of arbitrary dimension defined over a number field. Instead of using the method (P1)-(P3) we avoid embedded resolution of singularities by applying Newton's method, obtaining a much smaller set of candidate poles which turn out to induce eigenvalues of complex monodromy.

There are other classes of singularities where the embedded resolution is known. For example, for any hypersurface singularity defined by an analytic function which is non-degenerated with respect to its Newton polytope, problems (P1) and (P3) above are solved. Nevertheless the problem (P2) seems to be a hard combinatorial problem and was partially solved by F. Loeser adding some extra technical conditions [34].

Let us consider the simplest cases where f has non-isolated singularities, namely the case of homogeneous surfaces. In this case problems (P1) and (P3) are easily solved; using it B. Rodrigues and W. Veys proved in [43] the Monodromy Conjecture for any homogeneous polynomial $f_d \in \mathbb{C}[x_1, x_2, x_3]$ satisfying

$$\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) \neq 0. \tag{*}$$

For any degree d and any homogeneous polynomial $f_d \in \mathbb{C}[x_1, x_2, x_3]$ a candidate pole is $s_0 = -\frac{3}{d}$ (it can be seen when one blows up once at the origin) and then (P2) presents new difficulties. A sufficient condition for the candidate pole $s_0 = -\frac{3}{d}$ of $Z_{top,0}(f, s)$ to verify the Monodromy Conjecture is (*), and this is the reason why the above authors should add this condition. This problem has been solved in [13].

An embedded resolution is also known for superisolated surface singularities, SIS for short, see [3]. This class "contains" in a canonical way the theory of complex projective plane curves, which gives a series of nice examples and counterexamples. They were introduced by I. Luengo in [36] in order to show that the μ-constant stratum in the semiuniversal deformation space of an isolated hypersurface singularity, in general, is not smooth. Later E. Artal in [3] used them to provide a counterexample for S.S.–T. Yau's conjecture (showing that, in general, the link of an isolated hypersurface surface singularity and its characteristic polynomial do not determine the embedded topological type of the singular germ).

¿From now on, we will consider the case of surface singularities, i.e. $n = 2$. A hypersurface singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$ (where f_j is homogeneous of degree j) is superisolated if the projective plane curve $C_d := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. In this case the embedded topological type (and the equisingular type) of f does not depend on the choice of f_j's (for $j > d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j = 0$ for any $j > d + 1$ and $f_{d+1} = l^{d+1}$ where l is a linear form not vanishing at $\operatorname{Sing}(C_d)$.

The main achievement of [13] is to prove the Monodromy Conjecture for SIS, see Theorem 3.1. We will review some of the results contained in this paper and how they can be applied to study some arrangements of rational plane curves.

The germ $(V, 0) \subset (\mathbb{C}^3, 0)$ is an isolated surface singularity. Hence $H_0(F, \mathbb{C})$ and $H_2(F, \mathbb{C})$ are the only non-vanishing homology vector spaces on which the monodromy acts (we denote the Milnor fiber by F). The only eigenvalue of the action of the monodromy on $H_0(F, \mathbb{C})$ is equal to 1. The characteristic polynomial of the action of the complex monodromy on $H_2(F, \mathbb{C})$ is given by the formula

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C_d)}}{(t - 1)} \prod_{P \in \operatorname{Sing}(C_d)} \Delta^P(t^{d+1}),$$

where $\Delta^P(t)$ is the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C_d, P) on $H_1(F_{g^P}, \mathbb{C})$ and g^P denotes the local equation of C_d at P, see [3]. Its local topological zeta function satisfies the following equality, see [13, Corollary 1.12]:

$$\begin{aligned} Z_{top,0}(V, s) = {} & \frac{\chi(\mathbb{P}^2 \setminus C_d)}{t - s} + \frac{\chi(\check{C}_d)}{(t - s)(s + 1)} + \\ & + \sum_{P \in \operatorname{Sing}(C_d)} \left(\frac{1}{t} + (t + 1) \left(\frac{1}{(t - s)(s + 1)} - \frac{1}{t} \right) Z_{top,P}(g^P, t) \right), \end{aligned}$$

where $t := 3 + (d + 1)s$.

The following properties can be easily described from the previous equalities:

Proposition 2.2. *Let $\mathcal{P}$ be the set of poles of $Z_{top,0}(V, s)$.*

(i) $\mathcal{P} \subset \{-1, -\frac{3}{d}\} \cup \bigcup_{P \in \operatorname{Sing}(C_d)} \left\{ -\frac{3 + t_0}{(d + 1)} \mid t_0 \text{ pole of } Z_{top,P}(g^P, t) \right\}$.

(ii) *If $-\frac{3}{d} \neq s_0 \in \mathcal{P}$ then $\exp(2i\pi s_0)$ is an eigenvalue of the monodromy zeta function of V.*

(iii) *Let $s_0 = -\frac{3}{d}$. If s_0 is a pole of $Z_{top,P}(C_d, s)$ at some point $P \in \operatorname{Sing}(C_d)$ and either $\chi(\mathbb{P}^2 \setminus C_d) > 0$ or $\chi(\mathbb{P}^2 \setminus C_d) = 0$, then $\exp(2i\pi s_0)$ is an eigenvalue of the monodromy zeta function of V.*

(iv) *If* $s_0 = -\frac{3}{d}$ *is a multiple pole of* $Z_{top,0}(V,s)$ *then* $\exp(2i\pi s_0)$ *is an eigenvalue of the local monodromy zeta function at some singular point of* C_d.

(v) *If* $s_0 = -\frac{3}{d}$ *is not a pole of* $Z_{top,P}(C_d, s)$, *then the residue of* $Z_{top,0}(V,s)$ *at* $-\frac{3}{d}$ *equals* $d\rho(C_d)$ *where*

$$\rho(C_d) := \chi(\mathbb{P}^2 \setminus C_d) + \chi(\check{C}_d)\frac{d}{d-3} + \sum_{P \in \mathrm{Sing}(C_d)} Z_{top,P}(C_d, -\frac{3}{d}).$$

Following Proposition 2.2, the Monodromy Conjecture for SIS is proved in all but two cases:

(N-1) $\chi(\mathbb{P}^2 \setminus C_d) = 0$, $s_0 = -\frac{3}{d}$ is not a pole for the local topological zeta function at any singular point in C_d and $\rho(C_d) \neq 0$.

(N-2) $\chi(\mathbb{P}^2 \setminus C_d) < 0$.

Definition 2.3. We say that a degree d effective divisor D on $\mathbb{P}^2$ ($d > 3$) is a *bad divisor* if $\chi(\mathbb{P}^2 \setminus D) \leq 0$ and $s_0 = -\frac{3}{d}$ is not a pole of $Z_{top,P}(g_D^P, s)$, for any singular point P in its support D_{red}, where g_D^P is the local equation of the divisor D at P.

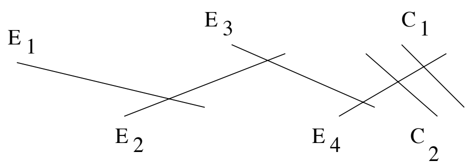
In the case of curves, if an exceptional divisor E_i satisfies $\chi(\check{E}_i) = 0$ ($\check{E}_i = E_i \setminus \bigcup_{j \neq i} E_j$) then E_i does not contribute to the candidate pole $\frac{-\nu_i}{N_i}$ of $Z_{top,0}(f,s)$. This question is more complicated in the case of surfaces. W. Veys proved in [44] for many such configurations that E does not contribute to the candidate pole $-\nu/N$, assuming that E doesn't intersect any other component with the same ratio of numerical data (this is the general case).

In [13], E. Artal, Pi. Cassou–Nogués, I. Luengo and A. Melle find that some candidate poles which appear only on exceptional divisors E_i of the resolution verifying $\chi(\check{E}_i) = 0$ are actual poles of the topological zeta function. This is the case for the first exceptional component of the resolution of a SIS singularity whose tangent cone D is a bad divisor with residue $\rho(D) \neq 0$ at the pole $-\frac{3}{d}$.

Example 2.4. Let $D \subset \mathbb{P}^2$ be the union of two smooth conics C_1 and C_2 which meet at only one point $\{P\} = C_1 \cap C_2$. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of the singularity of D at the point P. The rational surface X has the configuration of curves and the corresponding associated invariants shown in Figure 1.

Below we compute $Z_{top,0}(V_D, s)$ and $\Delta_V(t)$ for a SIS singularity $(V_D, 0) \subset (\mathbb{C}^3, 0)$ whose tangent cone is D. In this case $\chi(\mathbb{P}^2 \setminus D) = 0$, and $s_0 = -\frac{3}{4}$ is not a pole of $Z_{top,P}(D,s)$ for the germ of curve D at P. Hence D is a bad divisor on $\mathbb{P}^2$. Since the residue $\rho(D) \neq 0$, $s_0 = -\frac{3}{4}$ is a simple pole of $Z_{top,0}(V_D, s)$ and, as one can easily check, $\exp(-2i\pi\frac{3}{4})$ is a root of $\Delta_V(t)$.

$$Z_{top,P}(D,s) = \frac{3s+5}{(1+s)(5+8s)},$$



	(N_i, ν_i)	w_i	E_i^2
E_1	$(2,2)$	$\frac{1}{2}$	-2
E_2	$(4,3)$	0	-2
E_3	$(6,4)$	$-\frac{1}{2}$	-2
E_4	$(8,5)$	-1	-1
C_1, C_2	$(1,1)$	$\frac{1}{4}$	0

Figure 1: Minimal embedded resolution.

$$Z_{top,0}(V_D, s) = \frac{130s + 20s^2 + 87}{(1+s)(3+4s)(29+40s)},$$

$$\Delta_V(t) = \frac{(t^5-1)(t^{40}-1)}{(t^{10}-1)(t-1)}.$$

When the tangent cone C_d is a bad divisor, $s_0 = -\frac{3}{d}$ is a simple pole of $Z_{top,0}(f, s)$ if and only if $\rho(C_d) \neq 0$.

The main part of [13, §2] is devoted to determining bad divisors D on $\mathbb{P}^2$ such that $\rho(D) \neq 0$. Note that the Euler–Poincaré characteristic condition on a bad divisor D implies that D has at least two irreducible components, all of them rational curves, see [15]. Next result of [13] is equivalent to the Monodromy Conjecture for SIS.

Corollary 2.5. *Let D be a bad divisor of degree d on $\mathbb{P}^2$. If $\rho(D) \neq 0$ then D has only one singular point and* $\exp(2i\pi(-\frac{3}{d}))$ *is an eigenvalue of the complex monodromy at that singular point.*

2.2. Rational arrangements of plane curves

We explain now how to use the results of [13] to prove the non-existence of rational arrangements in $\mathbb{P}^2$. Therefore, we restrict ourselves to arrangements whose complement in the plane has null Euler–Poincaré characteristic.

Definition 2.6. An *rational arrangement* $D = \bigcup_{i=0}^{r} \mathcal{C}_i$ *of plane curves* is a reduced plane curve such that each irreducible component is rational. The dual graph of the minimal embedded resolution of D is determined by the following data:

(1) The degrees d_i of the irreducible components of D,

(2) The list of the topological types of the local singularities of D,

(3) The irreducible component of D which contains each branch Γ of D at a singular point.

We call these data *the combinatorial type* of the curve D in $\mathbb{P}^2$. We also call the data in (2) together with the total degree d of D *the local combinatorial data* of D in $\mathbb{P}^2$.

Given a divisor D on $\mathbb{P}^2$ and a point $P \in D$, the local topological zeta function $Z_{top,P}(D,s)$, the residue $\rho(D)$ and the eigenvalues of the complex algebraic monodromy of (D,P) are determined by the local combinatorial data of D. Hence the Corollary 2.5 gives necessary conditions on the local combinatorial data for D to exist.

Question. *Is there any set of combinatorial data for a possible, but non-existing, divisor D on $\mathbb{P}^2$ such that D is a bad divisor with $\rho(D) \neq 0$ satisfying the statement of Corollary (2.5) ?*

Let us present some few examples. We will use the standard Arnold notation for singularities, see e.g. in this volume Theorem 1.3 in [28].

Example 2.7. Let D consist of two conics which only meet at one point and a line which is tangent to each conic in different points. Using elementary properties of pencils of conics it is easy to see that D does not exist. In this case, the residue $\rho(D)$ would be $-3/5$ (different from 0) but there would be three singular points. Thus it would contradict to Corollary 2.5.

Example 2.8. Consider a rational curve C of degree six with only one singular point P which is a simple singularity. Then P can be either an $\mathbb{A}_{19}$ or $\mathbb{A}_{20}$ singularity. It is known that the $\mathbb{A}_{19}$ case exists, e.g. see [48]. The double covering of $\mathbb{P}^2$ ramified along C is a $K3$-surface. Using $K3$-surface theory one shows that the $\mathbb{A}_{20}$ case is not possible. Let's see how to prove this with our construction.

Assume such a curve exists. Let $D = C \cup C_2$ be the curve whose components are the sextic C with the $\mathbb{A}_{20}$ singularity at P and C_2, where the latter is the unique conic passing through the first five infinitely near points of C at P. We suppose that this conic in fact passes through the sixth infinitely near point of C at P. Hence the conic only meets C at its singular point. The residue $\rho(D)$ would be different from 0 and the characteristic polynomial of the monodromy of (D,P) would turn out to be

$$\Delta_{D,P}(t) = \frac{(t-1)(t^{17}-1)(t^{54}-1)}{(t^{27}-1)(t^3-1)}.$$

Hence D does not exist because $\exp(2i\pi(-3/8))$ is not an eigenvalue of the complex monodromy of D at P.

Example 2.9. Consider C a rational curve of degree 10 with only one singular point P whose multiplicity sequence is $[4,4,4,4,4,4,1,1,1,1] = [4_6]$, (this curve exists and it appears in the classification of H. Kashiwara, see [13, Appendix]).

Let $D = C \cup C_2$ be the curve whose components are C and C_2, where the latter is the unique conic passing through the first five infinitely near points of C at P. In this case the residue $\rho(D) = -3$ and $\exp(2i\pi(-3/12))$ is a root of the characteristic polynomial of the monodromy of D at P. Its Alexander polynomial is the following:

$$\Delta_{D,P}(t) = \frac{(t-1)(t^{25}-1)(t^{120}-1)}{(t^5-1)(t^{30}-1)}.$$

The following is a list of several possible cuspidal rational curves of degree 10 which might exist. We describe each singularity as a sequence of multiplicities.

$$\begin{array}{cccc}
[4_5,2_6], & [4_5,2_5]+1\mathbb{A}_2, & [4_5,2_4]+2\mathbb{A}_2, & [4_5,2_4]+1\mathbb{A}_4, \\
[4_5,2_3]+3\mathbb{A}_2, & [4_5,2_3]+3\mathbb{A}_2, & [4_5,2_3]+1\mathbb{A}_2+1\mathbb{A}_4, & [4_5,2_3]+1\mathbb{A}_6, \\
[4_5,2_3]+1\mathbb{A}_6, & [4_5,2_2]+4\mathbb{A}_2, & [4_5,2_2]+2\mathbb{A}_2+1\mathbb{A}_4, & [4_5,2_2]+2\mathbb{A}_4, \\
[4_5,2_2]+1\mathbb{A}_2+1\mathbb{A}_6, & [4_5,2_2]+1\mathbb{A}_2+1\mathbb{E}_6, & [4_5,3]+1\mathbb{A}_2+1\mathbb{A}_4, & [4_5,3]+1\mathbb{A}_2, \\
[4_5,3]+1\mathbb{A}_6, & [4_5,3]+1\mathbb{A}_6, & [4_5,2_2]+1\mathbb{A}_8. &
\end{array}$$

For each possible curve in the list, one considers the curve D as the union of such a curve of degree 10 and the conic as before, then anyl of them define a bad divisor with residue $\rho(D) \neq 0$. Thus all of them but the first one do not exist because they have more than one singular point. In fact, the first one would also give a counter-example to Corollary 2.5. The invariants for an SIS $(V,0)$ whose tangent cone is a curve with such properties are

$$Z_{top,0}(V,s) = \frac{93547584s^4 + 436242144s^3 + 294239146s^2 + 71173441s + 5854275}{7(1+s)(1+4s)(59+234s)(81+325s)(175+702s)},$$

$$\Delta_V(t) = \frac{(t^{819}+1)(t^{364}-t^{351}+t^{13}-1)(t^{260}+t^{195}+t^{130}+t^{65}+1)}{(t-1)}.$$

Therefore $s_0 = -1/4$ should be a pole of the topological zeta function but $-i = \exp(2i\pi\frac{-1}{4})$ is not an eigenvalue of the complex monodromy. Hence such a curve does not exist.

2.3. Superisolated singularities whose link is a rational homology sphere

In [38] L. Nicolaescu and A. Némethi formulated the following SWC-conjecture (as a generalization of the "Casson invariant conjecture" of Neumann and Wahl [41]): *If the link of a $\mathbb{Q}$-Gorenstein normal surface singularity $(V,0)$ is a rational homology sphere then*

$$p_g = \mathbf{sw}(M) - (K^2+s)/8.$$

Here, p_g is the geometric genus of $(V,0)$, $\mathbf{sw}(M)$ is the Seiberg–Witten invariant of the link M of $(V,0)$ associated with its canonical $spin^c$ structure, K is the canonical cycle associated with a fixed resolution graph Γ of $(V, 0)$, and s is the number of vertices of Γ (see [38] for more details).

The SWC-conjecture was verified succesfully for many different families, see e.g. [38, 39, 40]. But recently I. Luengo, A. Melle and A. Némethi in [37] found some counterexamples based on superisolated singularities. For an SIS $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$ (where f_j is homogeneous of degree j) with tangent cone the reduced projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ with isolated singularities $\{p_i\}_{i=1}^N$, its link M is a rational homology sphere if and only if C is a cuspidal rational curve, i.e. if all the germs (C, p_i) are locally irreducible. In the sequel we also will assume this fact.

In [37] the authors have shown that an SIS with $N = \#Sing(C) \geq 2$ tipically does not satisfy the above Seiberg–Witten invariant conjecture. On the other hand, even after an intense search of the existing cases, the authors were not able to find any counterexample with $N = 1$.

To understand the case $N = 1$, J. Fernández de Bobadilla, I. Luengo, A. Melle and A. Némethi have started the study of uni-cuspidal rational plane curves. Recall that the characteristic polynomial Δ of $(C, p) \subset (\mathbb{P}^2, p)$ is a complete (embedded) topological invariant of this germ, similarly as the semigroup $\Gamma_{(C,p)} \subset \mathbb{N}$ (generated by all the possible intersection multiplicities $i(\{g = 0\}, C)$ at p for all $g \in \mathcal{O}_{(\mathbb{P}^2,p)}$). By [16], $\Delta(t) = (1 - t) \cdot L(t)$, where $L(t) = \sum_{k\in\Gamma_{(C,p)}} t^k$ is the Poincaré series of $\Gamma_{(C,p)}$. This leads us to formulate the following conjecture, see [27].

Conjecture. The compatibility property (CP) of the semigroup. Assume that an irreducible rational plane curve C of degree d has only one singular point p, which is locally irreducible. Let $\Gamma_{(C,p)}$ be the semigroup of the germ (C, p). Then

$$\sum_{k\in\Gamma_{(C,p)}} t^{\lceil k/d\rceil} = \frac{1 - t^d}{(1 - t)^2}. \tag{CP}$$

The conjecture has been settled in many cases, see [27]. Let us review one of the casses. An irreducible plane curve C is said to be of *Abhyankar–Moh–Suzuki type* (*AMS type* for short) if there exists a line $L \subset \mathbb{P}^2$ such that $C \setminus L$ is isomorphic to $\mathbb{C}$. In our case $N = 1$, this means that $C \cap L = \{p\}$.

Not any rational curve with $N = 1$ is of *AMS* type, e.g. the curve in Example 2.9 is not. The simplest *AMS* curve is $\{zx^{d-1} + y^d = 0\}$. In this case $\Gamma_{(C,p)}$ is generated by two elements, $d - 1$ and d, and (CP) can be easily verified.

In fact (CP) is satisfied by any AMS curve. The idea is to identify $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus L$ and consider an algebraic automorphisms $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ with components (f, g). Recall, that by [1], a curve C is of *AMS* type if and only if it is the compactification in $\mathbb{P}^2$ of the zero locus of a component of a certain automorphism ϕ. The embedding of $\mathbb{C}^2$ into $\mathbb{P}^2$ allows us to view any automorphism of $\mathbb{C}^2$ as a birational transformation of $\mathbb{P}^2$. The point is that the combinatorics of the minimal embedded resolution of (C, p)

is closely related to the combinatorics of the minimal resolution of the indeterminacy of ϕ as a birational transformation of $\mathbb{P}^2$, and this last one can be described precisely. For details see the papers of J. Fernández de Bobadilla [25, 26]. Using these minimal resolutions the computation is carried on in a very combinatorial way.

3. Topology of rational arrangements of curves

Let $\mathcal{C} \subset \mathbb{P}^2$ be a projective curve whose irreducible components $\mathcal{C}_0, \mathcal{C}_1, ..., \mathcal{C}_r$ are rational curves . In this situation, we will call $\mathcal{C}$ a *rational arrangement*. For simplicity we will assume that $\mathcal{C}_0$ is a (non necessarily transversal) line so that one can naturally identify $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus \mathcal{C}_0$ and refer to $\mathcal{C}_{\text{af}} := \mathcal{C} \cap \mathbb{C}^2$ as the affine part of $\mathcal{C}$. In this survey, we will concentrate on the study of four of the main invariants of the topological pair $(\mathbb{P}^2, \mathcal{C})$. We will briefly describe them.

3.1. Cohomology ring

The ring structure of $H^*(X_\mathcal{C})$, $X_\mathcal{C} := \mathbb{P}^2 \setminus \mathcal{C}$ was first considered by Arnold for line and hyperplane arrangements. He proved its structure was determined by the *combinatorics* of the line arrangement in the special case of the discriminant hyperplane arrangement $\{x_i = x_j \mid i \neq j\}$. An abstract line combinatorics can be defined as a pair $(\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a finite set and $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$ satisfies:

(C1) For all $P \in \mathcal{P}$, $\#\mathcal{P} \geq 2$;

(C2) For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, $\exists! P \in \mathcal{P}$ such that $\ell_1, \ell_2 \in P$.

Further studies by Brieskorn and Orlik-Solomon generalized this result for any line arrangement as follows:

Theorem 3.1. *Let $\mathcal{L}$ be a line arrangement and $X_\mathcal{L} := \mathbb{P}^2 \setminus \mathcal{L}$, then $H^1(X_\mathcal{L})$ is a free group generated by the meromorphic forms $\sigma_i := \frac{d\ell_i}{2\pi\sqrt{-1}\ell_i}$ and $H^*(X_\mathcal{L})$ is a quotient of the graded exterior algebra $\bigwedge^\bullet H^1(X_\mathcal{L})$ given by the following presentation:*

$$H^*(X_\mathcal{L}) \cong \langle \sigma_i, \sigma_{jk} : \begin{array}{l} \sigma_i \wedge \sigma_j = \sigma_{ij}, \\ \sigma_i \wedge \sigma_{jk} = \sigma_{ij} \wedge \sigma_{kl} = 0 \\ \sigma_{i_1 i_2} + \sigma_{i_2 i_3} + \sigma_{i_3 i_1} = 0 \end{array} \rangle, \tag{3.1}$$

where $i, j, k \in \{1, ..., n\}$ and $\ell_{i_1} \cap \ell_{i_2} \cap \ell_{i_3} \neq \emptyset$.

In particular, this ring depends only on the combinatorics of the arrangement. We have proved a generalization of this result for rational arrangements. The cohomology ring of a rational arrangement is not a quotient of the graded exterior algebra $\bigwedge^\bullet H^1(X_\mathcal{L})$ anymore, but it still depends only on the combinatorics, see Definition 2.6. One has the following result:

Theorem 3.2. *Let $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_r$ be a rational arrangement where $\mathcal{C}_0$ is a transversal line at infinity. The set of local branches of $\mathcal{R}$ at P will be denoted by $\Delta_P(\mathcal{R})$. Then there exists a set of 1-forms*

$$v_1(\mathcal{R}) = \left\{\sigma_i := \frac{d\mathcal{C}_i}{\mathcal{C}_i}\right\}_{i\in\{1,...,r\}}$$

and a set of 2-forms

$$v_2(\mathcal{R}) = \{\psi_P^{\delta,\delta'}\}_{\substack{P\in \mathrm{Sing}(\mathcal{C}_i\cup\mathcal{C}_j),\\ \delta\in\Delta_P(\mathcal{C}_i),\delta'\in\Delta_P(\mathcal{C}_j)}} \cup \{\psi_\infty^{i,k_i}\}_{k_i\in\{1,...,d_i-1\}}, \quad i,j\in\{1,...,r\},$$

such that the ring $H^*(X_\mathcal{R};\mathbb{C})$ is generated by $v(\mathcal{R}) = v_1(\mathcal{R})\cup v_2(\mathcal{R})$. Moreover, the following is a complete system of relations

$$\sigma_i\wedge\sigma_j = \sum_{\substack{P\in\mathcal{C}_i\cap\mathcal{C}_j,\\ \delta\in\Delta_P(\mathcal{C}_i),\delta'\in\Delta_P(\mathcal{C}_j)}} (\delta,\delta')_P\ \psi_P^{\delta,\delta'} + d_j\sum_{k_i=1}^{d_i}\psi_\infty^{i,k_i} - d_i\sum_{k_j=1}^{d_j}\psi_\infty^{j,k_j},$$

$$\sigma_i\wedge\sigma_j\wedge\sigma_k = 0,$$

$$\psi_P^{\delta,\delta'} = -\psi_P^{\delta',\delta},$$

$$\psi_P^{\delta,\delta'}\wedge\varphi = \varphi\wedge\psi_P^{\delta,\delta'} = 0 \quad \forall\varphi\in v(\mathcal{R}),$$

$$\sigma_i\wedge\varphi = \varphi\wedge\sigma_i = 0 \quad \forall\varphi\in v_2(\mathcal{R})$$

and

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

$$\forall P\in\mathcal{C}_i\cap\mathcal{C}_j\cap\mathcal{C}_k, \delta_1\in\Delta_P(\mathcal{C}_i), \delta_2\in\Delta_P(\mathcal{C}_j) \text{ and } \delta_3\in\Delta_P(\mathcal{C}_k),$$

where $(\delta,\delta')_P$ denotes the multiplicity of intersection of δ and δ' at P.

3.2. Characteristic varieties

Definition 3.3. Let us denote by G the fundamental group of the complement of the curve $\mathcal{C}$ in $\mathbb{P}^2$. The quotient $M_\mathcal{C}^\mathbb{Z} := G'/G''$ has a natural structure of $\mathbb{Z}[H_1]$-module (where the action is given by conjugation) called *Alexander Invariant of $\mathcal{C}$*. Note that $\mathbb{Z}[H_1] = \mathbb{Z}[\mathbb{Z}^r]$. Tensoring both group ring and module by $\mathbb{C}$, $M_\mathcal{C} := M_\mathcal{C}^\mathbb{Z}\otimes\mathbb{C}$ becomes a Λ-module over the ring of Laurent polynomials ($\Lambda = \mathbb{C}[\mathbb{Z}^r] = \mathbb{C}[t_1^{\pm1},...,t_r^{\pm1}]$). The *$k$-th characteristic variety* is the support of the k-th exterior power of the module $M_\mathcal{C}$, that is, the set of prime ideals $\mathfrak{p}$ of Λ such that $\bigwedge^k M_\mathcal{C}$ becomes trivial when localizing at $\mathfrak{p}$. We will denote the k-th characteristic variety of $\mathcal{C}$ by $\mathrm{Char}_k(X_\mathcal{C})$ or simply by $\mathrm{Char}_k(\mathcal{C})$.

Note that $\mathrm{Char}_k(\mathcal{C})$ is a subvariety of

$$\operatorname{Spec}\Lambda \cong (\mathbb{C}^*)^r =: \mathbb{T}_r. \tag{3.2}$$

In fact, by defining $\mathrm{Char}_0(\mathcal{C}) = \mathbb{T}_r$, the sequence of characteristic varieties of $\mathcal{C}$ produces a stratification of $\mathbb{T}_r$ by considering their set of zeroes. Another stratification of $\mathbb{T}_r = \mathrm{Hom}(G, \mathbb{C}^*)$ (the space of rank one representations on $X_{\mathcal{C}}$) is given by the cohomology support locus $\Sigma_\bullet$ of $\mathcal{C}$, that is,

$$\Sigma_k(\mathcal{C}) = \{\rho \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H^1(X_{\mathcal{C}}; \mathbb{C}_\rho) \geq k\}.$$

E. Hironaka [29] and A. Libgober [32] have studied the relation between $\Sigma_k(X_{\mathcal{C}})$ and $\mathrm{Char}_k(\mathcal{C})$. Both stratifications coincide, as sets points, outside $\mathbb{1}_r = (1, 1, ..., 1)$. More specifically, $\Sigma_k(\mathcal{C}) \setminus \mathbb{1}_r = \mathrm{Char}_k(\mathcal{C}) \setminus \mathbb{1}_r$ as sets of points.

The structure of $\Sigma_k(\mathcal{C})$ is known to be a finite union of translated tori by torsion points [2]. The tangent cone of $\Sigma_k(\mathcal{C})$ at $\mathbb{1}_r$ has a combinatorial description by means of a classical object called Aomoto complex. Let $\omega \in H^1(X_{\mathcal{L}})$ be a 1-cochain. Note that Theorem 3.1 allows one to define a complex on $H^*(X_{\mathcal{L}})$ by multiplication by ω:

$$0 \quad \to \quad H^1(X_{\mathcal{L}}) \quad \stackrel{\wedge\omega}{\to} \quad H^2(X_{\mathcal{L}}) \quad \stackrel{\wedge\omega}{\to} \quad ...$$

Such a family of complexes is parametrized by $H^1(X_{\mathcal{L}})$, each complex is known as Aomoto complex and will be denoted by (A, ω). One can define the stratification

$$\mathcal{V}_k := \{\omega \in H^1(X_{\mathcal{L}}) \mid \dim_{\mathbb{C}} H^1(A, \omega) \geq k\}.$$

Each space $\mathcal{V}_k$ is known as *resonance variety of order k of $\mathcal{C}$*. Cohen-Suciu [19] and Libgober [32] have shown that $\mathcal{V}_k$ are the tangent cone of $\Sigma_k(\mathcal{L})$ at $\mathbb{1}_r$. The same holds for rational arrangements (Cogolludo [18]).

In order to study the components of $\mathrm{Char}_k(\mathcal{C})$ not passing through $\mathbb{1}_r$ one needs to distinguish two kinds of components. Consider the coordinate hypertorus $\mathbb{T}_r^i := \{(t_1, ..., t_r) \in \mathbb{T}_r \mid t_i = 1\}$. Any component of $\mathrm{Char}_k(\mathcal{C})$ contained in $\mathbb{T}_r^i$ is called *coordinate component*. The study of non-coordinate components is closely related to the study of position of singularities of a curve [32] and thus, rational arrangements with the same combinatorics might have different non-coordinate components [11].

Also, suppose that $\mathcal{C}' \subset \mathcal{C}$ is another curve obtained from $\mathcal{C}$ by removing an irreducible component, say $\mathcal{C}_i$. Then it is easy to see that $\mathrm{Char}_k(\mathcal{C}') \subset \mathbb{T}_r^i \cap \mathrm{Char}_k(\mathcal{C})$. If a component of $\mathrm{Char}_k(\mathcal{C})$ can be seen as $\mathrm{Char}_k(\mathcal{C}')$ for some $\mathcal{C}'$, then it is called *non-essential component*, otherwise it is an *essential component*. Hence non-essential components are coordinate components, but not conversely [8].

Question. *Is there any algebraic condition on the position of singularities of $\mathcal{C}$, for the existence of essential coordinate components of $\mathrm{Char}_k(\mathcal{C})$?*

3.3. Fundamental group

The fundamental group of the complement $X_{\mathcal{C}} := \mathbb{P}^2 \setminus \mathcal{C}$ of any algebraic curve is a finitely presented group [50] and a presentation can be obtained from the braid monodromy of $\mathcal{C}$. Fundamental groups of curves with only nodes as singularities is Abelian [20], rational curves with only cusps as singularities have a braid group as fundamental group [49] and simple examples of curves have free groups as fundamental groups, for example the fundamental group of the complement of $r+1$ incident lines is $\mathbb{F}_r$, the free group on r generators.

The first example of rational arrangements with the same combinatorics but non-isomorphic fundamental groups was published by Artal–Carmona in [5]. The arrangements involved were curves of degree 7 with three irreducible components: two smooth conics and a nodal cubic. Their Alexander polynomials being both trivial, the way to prove this result was directly checking that the groups were not isomorphic.

For some time, the question about the dependence of the fundamental group of line arrangements (or rational arrangements) only on their combinatorics had been open. In 1994, in a preprint, Rybnikov [42] claimed to have found an example of two line arrangements having the same combinatorics but non-isomorphic fundamental groups. Most probably due to the difficulty of verification this paper was never published. Keeping his main ideas and developing techniques on derived series and combinatorics, we were able to present a new proof of this very important result [10] which also leads to some new problems on combinatorics and fundamental groups. In particular, note that if $\mathcal{C}$ is a rational arrangement, then there is a canonical basis of $H_1(X_{\mathcal{C}})$ given by boundaries of small disks transversal to each irreducible component. Such generators are called *meridians*. So, in a way, a basis of $H_1(X_{\mathcal{C}})$ only depends on the combinatorics (in fact, only on the number of irreducible components) and hence we will denote it by H_1. Some automorphisms of H_1 come from "rearranging" the canonical basis, those automorphisms are called *geometric automorphisms* ($\mathrm{Geom}(H_1) \subset \mathrm{Aut}(H_1)$). Now assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two rational arrangements with the same combinatorics. Any isomorphism $\varphi : G_1 \to G_2$ (G_i the fundamental group of $X_{\mathcal{C}_i}$) induces an automorphism $\varphi_1 : H_1 \to H_1$, but not necessarily geometric. If the automorphisms are necessarily geometric, that is, if $\{\varphi_1 \in \mathrm{Aut}(H_1) \mid \varphi \in \mathrm{Isom}(G_1, G_2)\} \subset \mathrm{Geom}(H_1)$, then the combinatorics is called *homologically rigid*. The question arises

Question. *What combinatorics are homologically rigid?*

The Alexander Invariants $M^{\mathbb{Z}}_{\mathcal{C}_1}$ and $M^{\mathbb{Z}}_{\mathcal{C}_2}$ defined in Section 3.3 are modules over $\mathbb{Z}[H_1]$. Hence, if the combinatorics is homologically rigid, then any isomorphism $\varphi \in \mathrm{Isom}(G_1, G_2)$ induces a morphism of modules $\tilde{\varphi} : M_{\mathcal{C}_1} \to M_{\mathcal{C}_2}$. The modules $M_{\mathcal{C}_i}$ are finitely generated. Proving that $M_{\mathcal{C}_1}$ and $M_{\mathcal{C}_2}$ are non-isomorphic modules over $\mathbb{Z}[H_1]$ proves that G_1 and G_2 are not isomorphic. Note that $M_{\mathcal{C}_i}$ and H_1 are extremities of a short exact sequence having G_i/G_i'' in the middle; we say that $M_{\mathcal{C}_1}$ and $M_{\mathcal{C}_2}$ are strongly isomorphic if there exist an isomorphism which extends to G_i/G_i''.

In Rybnikov's example, the combinatorics is homologically trivial and the Alexander Invariants are not strongly isomorphic over $\mathbb{Z}[H_1]$, but they are strongly isomorphic over $\mathbb{C}[H_1]$.

3.4. Braid monodromy

Let $P \in \mathcal{C}_0 \setminus \mathrm{Sing}(\mathcal{C})$ and $\mathcal{H}_P$ be the pencil of lines in $\mathbb{P}^2$ having P as base point. The pencil $\mathcal{H}_P$ defines a map $\varphi : \mathbb{C}^2 \to \mathbb{C}$, and the image of a line ℓ in $\mathcal{H}_P$ will be refer to as the parameter of ℓ. Except for a finite number of lines $\mathcal{L} := \{\ell_1, ..., \ell_n\}$ in $\mathcal{H}_P$, any $\ell \in \mathcal{H}_P$ intersects $\mathcal{C}$ in as many as d different points, where d is the degree of $\mathcal{C}$. Such a pencil induces a map $\varphi : (\mathbb{C}^2 \setminus \bigcup \mathcal{L}, \mathcal{C}_{\mathrm{af}}) \to \mathbb{C} \setminus \{a_1, ..., a_n\}$, where a_i is the parameter of ℓ_i, which is a locally trivial fibration.

Let us consider $\ell \in \mathcal{H}_P \setminus \mathcal{L})$ and let us denote by a its parameter. This locally trivial fibration induces an action $\pi_1(\mathcal{C} \setminus \{a_1, ..., a_n\}, a) \to \mathbb{B}_{d-1}(\ell \cap \mathcal{C}_{\mathrm{af}})$ from the fundamental group of the base, on the group of automorphisms of the fiber ℓ preserving both $\ell \cap \mathcal{C}_{\mathrm{af}}$ and P (that is, fixing the complement of a *big enough* disk in ℓ), such group of automorphisms can be identified with the braid group on $\#\ell \cap \mathcal{C}_{\mathrm{af}} = d-1$ strings. Such a map is called the *generic braid monodromy* of $\mathcal{C}_{\mathrm{af}}$. Generic braid monodromies determine the homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}_{\mathrm{af}}$ ([31]) and recently, J.Carmona has proved this object to determine the isotopy class of $(\mathbb{P}^2, \mathcal{C})$ [17]. Also, as a converse, orientation-preserving-homeomorphisms of $\left(\mathbb{P}^2, \mathcal{C} \cap (\bigcup \mathcal{L})\right)$ are proved to preserve braid monodromies [7]. Some of these results can be extended to non-generic braid monodromies ([17, 7]). In particular, if $\mathcal{C}$ only admits one non-generic braid monodromy so that the non-generic lines in the pencil $\mathcal{H}_P$ are contained in $\mathcal{C}$, then one has that $\mathcal{C} \cap (\bigcup \mathcal{L}) = \mathcal{C}$ and hence the non-generic braid monodromy becomes an invariant of orientation-preserving-homeomorphisms. Such curves are called *fibered curves.*

Considering a special kind of bases of $\pi_1(\mathbb{C} \setminus \{a_1, ..., a_n\}, a)$ as in Figure 2,

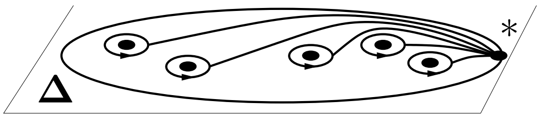


Figure 2: Geometric Basis

called *geometric basis*, braid monodromy can be regarded as an equivalence class of r-tuples of braids.

Braid monodromy can be used to distinguish embeddings of conjugated curves. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves with conjugated equations in a number field (that is, a Galois transformation on the number field sends one equation to the other). Therefore all the finite index normal subgroups of their fundamental groups coincide, and hence, the profinite completion of both groups coincide. The profinite completion of the fundamental group of an algebraic curve is also known as the *algebraic fundamental group*, π_{alg}. Hence, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are conjugated, then $\pi_{\mathrm{alg}}(X_{\mathcal{C}_1}) = \pi_{\mathrm{alg}}(X_{\mathcal{C}_2})$. In practice, all the effective topological invariants of curves only depend on the algebraic fundamental group. Another approach to study the embedding of a curve is to find invariants of the braid monodromy. A simple corollary of the main theorem in [7] states that,

Corollary 3.4. *Let $\mathcal{C}$ be a real fibered curve, then its non-generic braid monodromy is a topological invariant of $(\mathbb{P}^2, \mathcal{C})$.*

In [6] we use Corollary 3.4 to find two real conjugated arrangement of rational curves with non-homeomorphic pairs, that is, $(\mathbb{P}^2, \mathcal{C}_1) \not\approx (\mathbb{P}^2, \mathcal{C}_2)$. Recently, we use the same technique to find two real conjugated line arrangements with non-homeomorphic pairs [9]. Note that non-homeomorphic pairs does not imply non-isomorphic fundamental groups; in fact the complements may be isomorphic.

Problem. *Find an example of real conjugated line arrangements with non-isomorphic fundamental groups. More in general, give an example of curves with isomorphic algebraic fundamental group, but non-isomorphic fundamental groups.*

The braid monodromy method is also useful to understand the topological properties of tame polynomial mappings $f : \mathbb{C}^2 \to \mathbb{C}$, using the discriminant of the polar map associated to a generic direction. The author M. Escario has obtained results in two directions.

In the first one, he has been able to find tame polynomials (having a rational arrangement as special fiber) which have conjugate coefficients in a number field and which are not topologically equivalent; the so-called discriminant method allows to compute the Seifert form of the link at infinity in terms of a distinguished basis of vanishing cycles. With these data, it is possible to compute the monodromy action on the homology of the generic fiber and to check that they are not conjugated.

In the second one, using discriminant method and the work of Gabrielov [24] (aade la referencia que est abajo), it is possible to compute the intersection form of the Milnor fiber of an SIS in a distinguished basis of vanishing cycles. This new results provide new tools to study SIS and obtain new results concerning rational arrangements and rational cuspidal curves as it has been shown in sections 2.2 and 2.3.

References

[1] S.S. Abhyankar, T.T. Moh: *Embedding of line in the plane.* J. Reine angew. Math. **276** (1975), 148–166.

[2] D. Arapura: *Geometry of cohomology support loci for local systems I.* J. Algebraic Geom. **6** (1997), 563–597.

[3] E. Artal: *Forme de Jordan de la monodromie des singularités superisolées de surfaces.* Memoirs AMS **109** (1994), no. 525.

[4] ———: *Sur les couples de Zariski.* J. Algebraic Geom. **3** (1994), no. 2, 223–247.

[5] E. Artal, J. Carmona: *Zariski pairs, fundamental groups and Alexander polynomials.* J. Math. Soc. Japan **50** (1998), no.3, 521–543.

[6] E. Artal, J. Carmona, J.I. Cogolludo: *Effective invariants of braid monodromy.* Preprint 2002.

[7] ———: *Braid monodromy and topology of plane curves.* Duke Math. J. **118** (2003), no.2, 261–278.

[8] ———: *Essential coordinate components of characteristic varieties.* Math. Proc. Cambridge Philos. Soc. **136** (2004), no.2, 287–299.

[9] E. Artal, J. Carmona, J.I. Cogolludo, M.Á. Marco: *Topology and combinatorics of real line arrangements.* Preprint available at `arXiv:math.AG/0307296`, 2003.

[10] ———: *Invariants of combinatorial line arrangements and Rybnikov's example.* Preprint available at `arXiv:math.AG/0403543`, 2004.

[11] E. Artal, J. Carmona, J.I. Cogolludo, H.O Tokunaga: *Sextics with singular points in special position.* J. Knot Theory Ramifications **10** (2001), no.4, 547–578.

[12] E. Artal, Pi. Cassou–Nogués, I. Luengo, A. Melle: *Denef–Loeser zeta function is not a topological invariant.* J. London Math. Soc. (2) **65** (2002), no.1, 45–54

[13] ———: *Monodromy conjecture for some surface singularities.* Ann. Sci. École Norm. Sup. (4) **35** (2002), no.4, 605–640.

[14] ———: *Quasi-ordinary power series and theirs zeta-functions.* Preprint available at `arXiv:math.NT/0306249` 2003

[15] E. Artal, I. Luengo, A. Melle: *On a conjecture of W. Veys.* Math. Ann. **317** (2000), no.2, 325–327.

[16] A. Campillo, F. Delgado, S.M. Gusein–Zade: *On the monodromy of a plane curve singularity and the Poincaré series of the ring of functions on the curve. Functional Analysis and its Applications*, **33**(1) (1999), 56–67.

[17] J. Carmona: *Monodromía de trenzas y topología de curvas algebraicas.* Tesis Doctoral, Univ. de Zaragoza, 2003.

[18] J.I. Cogolludo: *Topological invariants of the complement to arrangements of rational plane curves.* Memoirs AMS. **159** (2002), no.756.

[19] D. Cohen, A. Suciu, *Characteristic varieties of arrangements.* Math. Proc. Cambridge Philos. Soc. **127** (1999), no.1, 33–53.

[20] P. Deligne: *Le groupe fundamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien.* Séminaire Bourbaki 32e année, **543** (1979), 1–10.

[21] J. Denef: *Report on Igusa's local zeta function.* Astérisque (1991), no. 201-203, Exp. no. 741, 359–386 (1992), Séminaire Bourbaki, Vol. 1990/91.

[22] J. Denef, F. Loeser: *Caractéristiques d'Euler–Poincaré, fonctions zêta locales et modifications analytiques.* Journal AMS **5** (1992), no.4, 705–720.

[23] ———: *Motivic Igusa zeta functions.* J. Algebraic Geom. **7** (1998), no.3, 505–537.

[24] A.M. Gabrièlov: *Polar curves and intersection matrices of singularities.* Invent. math. **54** (1979), no.1, 15–22.

[25] J. Fernández de Bobadilla: *A new geometric proof of Jung's Theorem on factorisation of automorphisms of* $\mathbb{C}^2$. To appear in Proc. Amer. Math. Soc.

[26] ———: *Moduli spaces of polynomials in two variables.* to appear in Memoirs AMS

[27] J. Fernández de Bobadilla, I, Luengo, A. Melle, A. Némethi: *in preparation.*

[28] J. F. Fernando, J. M. Ruiz: *Determinación finita en anillos de series formales.* In this volume.

[29] E. Hironaka: *Alexander stratifications of character varieties.* Ann. Inst. Fourier (Grenoble) **47** (1997), no.2, 555–583.

[30] A.J. de Jong, J. Steenbrink: *Proof of a conjecture of W. Veys.* Indag. Math. (N.S.) **6** (1995), no.1, 99–104.

[31] A. Libgober: *On the homotopy type of the complement to plane algebraic curves.* J. reine angew. Math. **367** (1986), 103–114.

[32] ———: *Characteristic varieties of algebraic curves.* Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 215–254, Kluwer Acad. Publ., Dordrecht 2001.

[33] F. Loeser: *Fonctions d'Igusa p-adiques et polynômes de Bernstein.* Amer. J. Math. **110** (1988), no.1, 1–21.

[34] ———: *Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton.* J. reine angew. Math. **412** (1990), 75–96.

[35] E. Looijenga: *Motivic measures.* Séminaire Bourbaki, Vol. 1999/2000. Astérisque No. **276** (2002), 267–297.

[36] I. Luengo: *The μ-constant stratum is not smooth.* Invent. math. **90** (1987), no.1, 139–152.

[37] I. Luengo, A.Melle, A. Némethi: *Links and analytic invariants of superisolated singularities.* Preprint available at `arXiv:math.AG/0312416;` to appear in J. Algebraic Geom..

[38] A. Némethi, L.I. Nicolaescu: *Seiberg–Witten invariants and surface singularities.* Geometry and Topology, Volume **6** (2002), 269–328.

[39] ———: *Seiberg–Witten invariants and surface singularities II (singularities with good* $\mathbf{C}^*$*-action).* Preprint available at `arXiv:math.AG/0201120;` to appear in Journal of LMS.

[40] ———: *Seiberg–Witten invariants and surface singularities III (splicings and cyclic covers).* Preprint available at `arXiv:math.AG/0207018.`

[41] W. Neumann, J. Wahl: *Casson invariant of links of singularities. Comment. Math. Helv.* **65** (1991), 58-78.

[42] G. Rybnikov: *On the fundamental group of the complement of a complex hyperplane arrangement.* Preprint available at `arXiv:math.AG/9805056`.

[43] B. Rodrigues, W. Veys: *Holomorphy of Igusa's and topological zeta functions for homogeneous polynomials.* Pacific J. Math. 201 (2001), no.2, 429–440.

[44] W. Veys: *Poles of Igusa's local zeta function and monodromy*. Bull. Soc. Math. France **121** (1993), no.4, 545–598.

[45] ———: *Determination of the poles of the topological zeta function for curves*. Manuscripta Math. **87** (1995), no.4, 435–448.

[46] ———: *Zeta functions for curves and log canonical models*. Proc. London Math. Soc. (3) **74** (1997), no.2, 360–378.

[47] W. Veys: *Structure of rational open surfaces with non-positive Euler characteristic*. Math. Ann. **312** (1998), 527–548.

[48] H. Yoshihara: *On plane rational curves*. Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no.4, 152–155.

[49] O. Zariski: *On the Poincaré group of rational plane curves*. Amer. J. Math **58** (1936), 607–617.

[50] ———: *A theorem on the Poincaré group of an algebraic hypersurface*. Annals of Math. (2) **38** (1937), no.1, 131–141.